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EXISTENCE IN THE LARGE FOR RIEMANN PROBLEMS FOR SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. An existence theorem in the large is obtained for the Riemann problem for nonlinear systems of conservation laws. Our principal assumptions are strict hyperbolicity, genuine nonlinearity in the strong sense, and the existence of a convex entropy function. The entropy inequality is used to obtain an a priori estimate of the strengths of the shocks and refraction waves forming a solution; existence of such a solution then follows by an application of finite-dimensional degree theory. The case of a single degenerate field is also included, with an additional assumption on the existence of Riemann invariants.

1. Main theorem. We consider initial-value problems for nonlinear systems of conservation laws of the form

$$u_t + f(u)_x = 0, \quad -\infty < x < \infty, t > 0, \tag{1.1}$$

u, f vectors of dimension n, f a smooth function of u, with given initial data of special form

$$u(x,0) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{1.2}$$

Problems of this form are often referred to as Riemann problems, and have been extensively studied. For u_-, u_+ sufficiently close together, the existence and uniqueness (in the small) of a solution satisfying the entropy condition is proved in [**3**] under relatively general conditions on the function f and without restriction on n. More recently, numerous results in the large have been obtained for the case $n = 2$ [**1, 2, 6, 12, 13, 14, 16**], and for the particular example of the Euler equations of gas dynamics [**5, 15, 17**]. Our objective here is to prove existence in the large for arbitrary values for n. In comparison with these previous results, this result is perhaps best described as an extension of the existence result of [**3**] to the large for genuinely nonlinear systems admitting an entropy function.

The following specific assumptions are made.

(i) The system (1.1) is strictly hyperbolic, i.e. the eigenvalues of $f_u(u)$ are real and distinct for all $u \in R^n$. We number the eigenvalues $\lambda_1(u), \lambda_2(u), \ldots, \lambda_n(u)$, in increasing order as usual.

(ii) The system (1.1) is genuinely nonlinear in the strong sense [**8**], i.e. the speed of any shock is not a characteristic speed of either of the two states connected by the shock.

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(iii) The system (1.1) admits a strictly convex entropy function $U(u)$ [**4**], i.e. there exist a pair of scalar functions U, F such that

$$F_u^T(u)U_u(u) = F_u(u) \quad \text{for all } u \in R^n. \tag{1.3}$$

Without loss of generality we assume $U(u) \geq 1$ for all u, making a transformation of the form

$$\begin{aligned} U(u) &\to U(u) - U(u_0) - (u - u_0)U_u(u_0) + 1, \\ F(u) &\to F(u) - U_u(u_0)f(u), \end{aligned} \tag{1.4}$$

if necessary, with some arbitrary fixed u_0. In addition to strict convexity, we assume that $\|U_u(v)\| \to \infty$ as $\|v\| \to \infty$.

(iv) For any fixed u_0, let $S(u_0)$ be the set of all states that can be connected to u_0 by a shock, i.e. for which the Rankine-Hugoniot relations are satisfied. Under assumption (ii) it is known [**8**] that $S(u_0)$ is composed of $2n$ one-manifolds, each beginning at u_0 and extending to infinity. Our assumption is that if $\{u_m\}_{m=1}^{\infty}$ is a sequence within $S(u_0)$ such that $\|u_m\| \to \infty$ as $m \to \infty$, then

$$|-s(u_0, u_m)(U(u_m) - U(u_0)) + F(u_m) - F(u_0)| \to \infty \text{ and } m \to \infty, \tag{1.5}$$

where $s(u_0, u_m)$ is the speed of the shock connected u_0 with u_m.

(v) For any fixed u_0, the set of states $R_k(u_0)$ which can be connected to u_0 by a k-rarefraction wave is a one-manifold extending to infinity in both directions and of course including the point u_0. (This also follows from assumptions (i) and (ii), as described in [**8**].) If $\{u_m\}_{m=1}^{\infty}$ is any sequence within $R_k(u_0)$ such that $\|u_m\| \to \infty$ as $m \to \infty$, then we assume

$$|\lambda_k(u_m) - \lambda_k(u_0)| \to \infty \quad \text{as } m \to \infty.$$

Our main theorem is then

THEOREM 1. *Under assumptions* (i)–(v), *there exists an entropy solution of the problem* (1.1), (1.2).

In general the solution consists of n distinct waves, each of which is a shock satisfying the entropy condition or a rarefraction wave. Of course, some of these waves may disappear, i.e. have amplitude zero. We note that for a genuinely nonlinear system in the sense of [**8**], the entropy conditions of [**3, 4**] are equivalent in the large, so no ambiguity exists in this respect. In §4, we show how this result can be extended to the case of a single degenerate field.

2. Proof of Theorem 1. For $0 \leq \xi \leq 1$, let $w(\xi) = (1 - \xi)u_- + \xi u_+$. For any fixed ξ, set $v_0 = u_-, v_n = w(\xi)$. We seek a set of $n - 1$ points $v_1, v_2, \ldots, v_{n-1}$, such that for each $k = 1, 2, \ldots, n$, v_{k-1} is connected on the left to v_k and on the right by either a k-shock satisfying the entropy condition or a k-rarefraction wave.

We first obtain an a priori estimate of the strengths of the various waves in such a solution. Suppose wave k, connecting v_{k-1} to v_k, is a k-refraction wave. From (1.3) it follows that [**4**]

$$U_t + F_x = 0 \tag{2.1}$$

in any region of the $x - t$ plane not containing shocks. Integrating (2.1) over the triangle bounded by the lines $x/t = \lambda_k(v_{k-1})$, $x/t = \lambda_k(v_k)$ and $t =$ constant, we find

$$F(v_k) = F(v_{k-1}) = \lambda_k(v_k)U(v_k) - \lambda_k(v_{k-1})U(v_{k-1}) - G_k; \tag{2.2}$$

$$G_k = \int_{\lambda_k(v_{k-1})}^{\lambda_k(v_k)} U(v(x/t))d(x/t) \geq 0, \tag{2.3}$$

where $v(x/t)$ is the value of v at points within the rarefraction wave corresponding to values of x/t between $\lambda_k(v_{k-1})$ and $\lambda_k(v_k)$. Next suppose that wave k is a shock. The entropy inequality satisfied by v_{k-1}, v_k is then [**4, 9**]

$$-s(v_k, v_{k-1})(U(v_k) - U(v_{k-1})) + F(v_k) - F(v_{k-1}) = E_k; \tag{2.4}$$

$$E_k = -\int_{\lambda_k(v_k)}^{s(v_k, v_{k-1})} [U_u(v(s))(v(s) - v_k) - U(v(s)) + U(v_k)]\, ds, \tag{2.5}$$

in which $v(s)$ is the point in $S(v_k)$ which can be connected to v_k by a shock of speed s. For a genuinely nonlinear system in the sense of [**8**], this point is unique [**10, 11**], and the shock curves can readily be parametrized in this manner. It is clear from $s(v_k, v_{k-1}) > \lambda_k(v_k)$ and the convexity of U that $E_k < 0$, as required by the entropy condition of [**4**].

To write (2.2) and (2.4) as a single equation, we set $E_k = 0, s_k^- = \lambda_k(v_{k-1})$, $s_k^+ = \lambda(v_k)$ for a rarefraction wave k, and $G_k = 0$, $s_k^- = s_k^+ = s(v_k, v_{k-1})$ for wave k a shock. Then either (2.2) or (2.4) assumes the form

$$F(v_{k-1}) - F(v_k) = s_k^- U(v_{k-1}) - s_k^+ U(v_k) - E_k + G_k; \tag{2.6}$$

summing (2.6) from $k = 1$ to $k = n$, we easily obtain

$$\begin{aligned} F(v_0) - F(v_n) &= \sum_{k=1}^{n} s_k^- U(v_{k-1}) - s_k^+ U(v_k) - E_k + G_k \\ &= s_1^- U(v_0) - s_n^+ U(v_n) + \sum_{k=1}^{n-1} (s_{k+1}^- - s_k^+) U(v_k) \\ &\quad + \sum_{k=1}^{n} (G_k - E_k). \end{aligned} \tag{2.7}$$

In (2.7) we replace v_0 by u and v_n by $w(\xi)$, use the positivity of $U(\cdot)$ in the first sum and the following lemmas, the proofs of which are deferred to §3:

LEMMA 1. *$s_{k+1}^- > s_k^+$ for all k.*

LEMMA 2. *There exist constants $c_1, c_2(\xi)$ such that*

$$s_1^- U(u_-) - \tfrac{1}{2}E_1 \geq c_1 \quad \textit{and} \quad - s_n^+ U(w(\xi)) - \tfrac{1}{2}E_n \geq c_2(\xi). \tag{2.8}$$

Using these results in (2.7) we obtain

$$\sum_{k=1}^{n} (G_k - E_k) < 2[F(u_-) - F(w(\xi)) - c_1 - c_2(\xi)] \leq K, \quad 0 \leq \xi \leq 1, \tag{2.9}$$

for a suitably chosen constant K. This is the required estimate on the strengths of the shock and rarefraction waves appearing in a solution.

Next we introduce a mapping $A: R^n \to R^n$. Given $\alpha \in R^n, \alpha = (\alpha_1, \ldots, \alpha_n)^T$, we evaluate $v_n = A(\alpha)$ as follows: set $v_0 = u_-$ and, successively for $k = 1, 2, \ldots, n$, find v_k connected on the right to v_{k-1} and on the left by a k-shock or a k-rarefaction wave, whose type and strength are determined by

$$
\begin{aligned}
G_k &= \alpha_k, \quad \alpha_k > 0 \text{ (wave } k \text{ is a rarefaction wave);} \\
v_k &= v_{k-1}, \quad \alpha_k = 0 \text{ (wave } k \text{ disappears);} \\
E_k &= -\mu_b(-\alpha_k), \quad \alpha_k < 0 \text{ (wave } k \text{ is a shock).}
\end{aligned} \tag{2.10}
$$

In (2.10), μ is a smooth monotone increasing function depending on the positive parameter b such that $\mu_b(K) = K$ for all b and $\mu_b(a) = ba^3$ for $|a|$ sufficiently small. In each application of (2.10), b is determined depending on v_{k-1} and on k but not on α_k, such that the mapping $\alpha_k \to v_k$, with v_{k-1} fixed, is of class C^1 at $\alpha_k = 0$.

The mapping A so determined is then of class C^1, piecewise smooth. A solution of the problem (1.1), (1.2) corresponds to a value of α such that $A(\alpha) = u_+$. We also note that the mapping A is defined on all of R^n; it is only here that assumptions (iv) and (v) are needed.

Let B denote the open hypercube in R^n, centered at the origin, sides parallel to the axes and each of length $2K$, and consider $\deg(A, B, w(\xi))$, where $\deg(\, , \,)$ denotes the finite-dimensional toplogical degree. From (2.9) it is clear that this quantity is defined and is independent of ξ for $0 \leq \xi \leq 1$. Thus to complete the proof it suffices to show that $\deg(A, B, u_+) = \deg(A, B, w(1)) = \deg(A, B, w(0)) = \deg(A, B, u_-)$ is not zero. To evaluate $\deg(A, B, u_-)$ we consider solutions of $A(\alpha) = u_-$. Obviously $\alpha = 0$ is one solution; from the assumption of strict hyperbolicity, it follows [**3**] that the contribution of this solution to $\deg(A, B, u_-)$ is either $+1$ or -1, depending on the normalization of the eigenvectors of $f_u(u_-)$. We claim that there can be no other solutions; otherwise there exists a nontrivial solution to this "trivial" Riemann problem, with $u_+ = u_-$. The nontrivial solution must satisfy

$$
U_t + F_x \leq 0, \tag{2.11}
$$

in the sense of distributions. But this is impossible, as readily seen by integrating (2.11) over a sufficiently wide rectangle in the $x - t$ plane, and taking u_- to be the point at which $U(\cdot)$ achieves its global minimum, i.e. setting $u_0 = u_-$ in (1.4) if necessary.

The single-valuedness of the solution so obtained follows from Lemma 1.

3. Proofs of the lemmas.

PROOF OF LEMMA 1. If both waves k and $k+1$ are rarefractions, then $s_k^+ = \lambda_k(v_k) < \lambda_{k+1}(v_k) = s_{k+1}^-$. If wave k is a shock of speed s and wave $k+1$ is a rarefraction, then since the system is assumed to be strongly nonlinear [**8**] we have $\lambda_k(v_k) < s < \lambda_{k+1}(v_k), s = s_k^+$ and $s_{k+1}^- = \lambda_{k+1}(v_k)$. Similarly, if wave k is a rarefraction and wave $k+1$ a shock of speed s, then $\lambda_{k+1}(v_k) > s > \lambda_k(v_k), s = s_{k+1}^-$ and $s_k^+ = \lambda_k(v_k)$. Thus the only delicate case arises when both waves k and $k+1$ are shocks, for then both s_k^+ and s_{k+1}^-, the speeds of the two shocks, lie between $\lambda_k(v_k)$ and $\lambda_{k+1}(v_k)$.

In this case v_{k-1} lies in the set of states that can be connected to v_k by a k-shock. For every $s \in (\lambda_k(v_k), s(v_{k-1}, v_k)]$ there exists a point $v(s)$ in this set, for which $s(v_k, v(s)) = s$. Similarly, v_{k+1} lies in this set of states that can be connected to v_k by a $(k+1)$-shock, and for every $s \in [s(v_{k+1}, v_k), \lambda_{k+1}(v_k))$ there exists a point which can be connected to v_k by a $(k+1)$-shock of speed s. Thus if $s(v_k, v_{k-1}) \geq s(v_{k+1}, v_k)$, equivalently $s_k^+ \geq s_{k+1}^-$, there exists at least one speed s_0 for which two different states can be connected to v_k by a shock of speed s_0, or else the set of states which can be connected to v_k by a k-shock and the set that can be connected by a $(k+1)$-shock coincide. Neither of these situations is permitted in strictly hyperbolic, strongly nonlinear systems of the form (1.1) [**8, 10, 11**].

PROOF OF LEMMA 2. The two statements are proved in exactly the same way, so we prove only the second. Fix $v_n = w(\xi)$. If wave n is a rarefaction, then $s_n^+ = \lambda_n(w(\xi)), E_n = 0$ and the result is obvious. Thus we consider wave n to be a shock, in particular a strong shock with large speed, for this is the only way that (2.8) could fail to hold.

We introduce $z = U_u, q(z) = u \cdot U_u - U$; it is easily verified that $q_z = u$ and $q_{zz} = U_{uu}^{-1}$, so that U strictly convex in u implies q strictly convex in z. In terms of the variable z, (1.1) assumes symmetric form [**10**]. Setting $U = z \cdot q_z - q$ in (2.5), we obtain the alternate expression for E_n,

$$E_n = -\int_{\lambda_n(w(\xi))}^{s(v_{n-1}, w(\xi))} [q(z(s)) - q(z_0) - q_z(z_0) \cdot (z(s) - z_0)]\, ds, \tag{3.1}$$

in which $z(s) = U_u(u(s))$ and $z_0 = U_u(w(\xi))$.

Now let v_{n-1} assume a sequence of values such that $s(v_{n-1}, w(\xi)) \to +\infty$. Since $s(v_{n-1}, w(\xi)) < \lambda_n(v_{n-1})$, it follows that $\lambda_n(v_{n-1}) \to \infty$, and thus, from the continuity of f_u, that $\|v_{n-1}\| \to \infty$. From assumption (iii), it follows that for large positive values of s, $\|z(s)\|$ and thus the integrand in (3.1) increase without limit, using the strict convexity of q. It follows that $E_n / s(v_{n-1}, w(\xi)) \to -\infty$ as $s(v_{n-1}, w(\xi)) \to +\infty$, so the existence of $c_2(\xi)$ satisfying (2.8) is clear.

4. The case of a single degenerate field.

The above results can be extended to include the case in which a single field, say the jth, is linearly degenerate.

THEOREM 2. *Suppose that the system* (1.1) *is such that the jth field is linearly degenerate, i.e.*

$$r_j(u) \cdot \operatorname{grad} \lambda_j(u) = 0 \ \textit{for all } u, \tag{4.1}$$

where $r_j(u)$ is the jth right eigenvector of $f_u(u)$ and the gradient in (4.1) *is with respect to u. Suppose that assumptions* (i) *and* (iii) *hold, and that assumptions* (ii), (iv) *and* (v) *are true for waves in all fields except the jth. Suppose further that there exist $n-1$ independent j-Riemann invariants $\gamma_1(u), \ldots, \gamma_{n-1}(u)$ so that two states v_-, v_+ can be connected by a contact discontinuity* (*a discontinuity in the j-field*) *if and only if*

$$\gamma_k(v_-) = \gamma_k(v_+), \quad k = 1, 2, \ldots, n-1. \tag{4.2}$$

Then there exists an entropy solution to the Riemann problem (1.1), (1.2).

PROOF. Then a priori estimate (2.9) remains valid in this case, but $G_j = E_j = 0$, so that we have no control over the strength of a contact discontinuity in a solution. We introduce the mapping $Y_\xi \colon R^{n-1} \to R^{n-1}$, with $Y_\xi(y), y = (y_1, \ldots, y_{n-1})^T$, evaluated as follows. We set $v_0 = u_-$ and find $v_1, v_2, \ldots, v_{j-1}$ using the first $j-1$

components of y as done for the mapping A, i.e. with y_k replacing α_k in (2.10). Next we set $v_n = w(\xi)$ and work backwards, replacing $\alpha_n, \alpha_{n-1}, \ldots, \alpha_{j+1}$ in (2.10) by $y_{n-1}, y_{n-2}, \ldots, y_j$ respectively and thus finding $v_{n-1}, v_{n-2}, \ldots, v_j$ successively. In general, of course, nothing can be said about the connecting of v_{j-1} and v_j obtained in this manner. However, we take

$$(4.3)\quad Y_\xi(y) = (\gamma_1(v_{j-1}) - \gamma_1(v_j), \gamma_2(v_{j-1}) - \gamma_2(v_j), \ldots, \gamma_{n-1}(v_{j-1}) - \gamma_{n-1}(v_j))^T.$$

From (4.2), (4.3) it is clear that a solution of the problem (1.1), (1.2) corresponds to a value of y such that $Y_1(y) = 0$. Let B' be the open hypercube in R^{n-1} centered at the origin, sides parallel to the axes and each of length $2K$. Then $\deg(Y_\xi, B', 0)$ is defined and independent of ξ for $0 \leq \xi \leq 1$. For $\xi = 0$, we again have only the trivial solution $y = 0$ of $Y_0(y) = 0$, and using the independence of the γ_k it follows that the contribution of this solution to the degree either $+1$ or -1. (The precise requirement of independence of the γ_k which is required is that the matrix $r_i(u_-) \cdot \operatorname{grad} \gamma_k(u_-)$ be nonsingular, where i takes on all integer values from 1 to n except for j, and $k = 1, 2, \ldots, n-1$.) Thus $\deg(Y_1, B', 0)$ is not zero, and the proof is complete.

The application of Theorem 2 to the equations of nonisentropic gas dynamics is complicated by the possible occurrence of the vacuum state. For this system, a state corresponding to positive density and pressure can be connected to the vacuum state by a rarefaction wave with a finite value of G_1, or G_3, in the present notation. Thus the mapping Y_ξ, as described above, is not defined on all of R^2.

At least in the ideal, polytropic case, the other assumptions are satisfied, so that Theorem 2 can be maintained by simply modifying the definition of Y_ξ for y_1 or y_2 above the limit imposed by the vacuum condition. An existence theorem is then recovered, but the solution so obtained may include the vacuum state (or some artificial state introduced in the modification of Y_ξ). Thus as in [**15**], in practice we shall require an explicit assumption on the data u_+, u_- to exclude the possible occurrence of the vacuum state in the obtained solution.

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